

JUMP CONDITIONS AND ENTROPY SOURCES IN TWO-PHASE SYSTEMS. LOCAL INSTANT FORMULATION

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Abstract—The continuum mechanics of two-phase systems involving surface tension and surface properties is discussed.

The integral forms of the balance laws are given for the following quantities: mass, linear momentum, angular momentum, total energy and entropy. Starting from these integral balance laws, the jump conditions and the entropy source at the interface are derived.

1. INTRODUCTION

In continuum mechanics the global conservation laws are reduced to local partial differential equations if one considers a point which does not belong to a surface of discontinuity. These partial differential equations express the local balances of the following fundamental quantities: mass, linear momentum, angular momentum, total energy and entropy. Whenever one considers a point belonging to a surface of discontinuity (Truesdell 1960) the local balance laws are no longer formulated in terms of partial differential equations, but in terms of jump conditions which relate the values of the fundamental quantities on both sides of the interface.

Jump conditions are of the utmost importance in the study of two-phase systems composed of fluid–fluid, fluid–solid or solid–solid couples. This latter case is seldom quoted but some authors recently used jump conditions in some problems of strength resistance (Freund 1970) or solid state physics (Ghez 1966).

The interest in studying jump conditions in two-phase systems is 3 fold:

- Jump conditions give the boundary conditions in a correct and straightforward way for the problems which involve a dynamical phase change.
- Jump conditions provide some relations between the interaction terms arising in two fluid models of two-phase systems (Delhaye 1968a, b; Vernier & Delhaye 1968; Ishii 1971; Kocamustafaogullari 1971; Bouré 1972, 1973a, 1973b; Réocreux 1973).
- Jump conditions enable the derivation of the local interfacial entropy source.

This local interfacial entropy source is itself of an utmost importance:

- It formulates the second law of thermodynamics. According to this law the entropy source has to be positive or null. Therefore, it is easy to judge the soundness of a two-phase flow model by deriving the entropy generation rate and by checking if the result is positive or null.
- It may give a formulation of the constitutive laws because it can be expressed as a bilinear function of thermodynamic fluxes and forces.
- It enables to put some restrictions on the proposed constitutive laws (Fisher & Leitman 1970).

Two kinds of jump conditions can be derived: primary jump conditions and secondary ones.

- The *primary jump conditions* are directly derived from the global balance laws written for the following fundamental quantities: mass, linear momentum, angular momentum, total energy, entropy.
- The *secondary jump conditions* are derived from the primary ones. They are written for the following quantities: mechanical energy, internal energy, enthalpy, entropy.

The above jump conditions reduce to the shock equations obtained either in inviscid fluid (Serrin 1959) or in weakly dissipative fluid (Germain 1960, 1961, 1964).

Phase change, surface tension and surface material properties have partially appeared in jump conditions for several years. Landau & Lifschitz (1959) and Wehausen & Laitone (1960) gave the momentum condition without any demonstration, in the case where there is neither phase change nor surface material properties. Scriven (1960) gave the first a derivation of a momentum jump condition which took into account surface tension and surface material properties but which neglected the phase change. Independently, Ghez & Puiz (1963) gave a different derivation but obtained the same result as Scriven (1960). Ghez (1964) derived an internal energy jump condition which seems to be erroneous. Slattery (1964) brought some corrections to the derivation of Scriven (1960) and obtained the linear and angular momentum jump conditions taking into account phase change, surface tension and surface material properties. This derivation was improved a few years later (Slattery 1967, 1968). Ghez (1966) found the energy jump condition and the entropy source in the same conditions as Slattery (1964). The entropy source calculated by Ghez (1966) was irrelevant but was corrected in a later publication (Ghez 1970). Independently from Ghez (1966), Delhaye (1968b) derived an energy jump condition which took into account a phase change and surface tension. Likht (1969) gave the same condition but in a wrong form. Drew (1971) gave the primary jump conditions under a global form, i.e. extended to a portion of interface. In his derivation, Drew (1971) took into account the phase change, surface tension and surface material properties. In the book by Barrère & Prud'homme (1973), the results of Ghez (1966, 1970) were recovered with an original method which entirely ignored the common mathematical works on surface theory (Weatherburn 1927; McConnell 1957; Aris 1962).

In the case where the surface tension is not taken into account, Standart (1967a; 1968)

gave an expression for the entropy source, also obtained by Campion & Lemaigre (1970). Fer (1971) found a similar formulation for reactive two-phase systems. The particular case of annular flows was examined by Giot & Fritte (1972).

In this paper, we derive the jump conditions in the case of two-phase flows involving surface tension and surface properties. The integral forms of the balance laws are given for the following quantities: mass, linear momentum, angular momentum, total energy and entropy.

Starting from these global balance laws, we derive the *primary* local laws (directly deduced from the global laws) and the *secondary* local laws (deduced from the primary local laws). Then follows the derivation of the entropy source.

The local laws are derived from the global ones with the help of a method which was already used by Truesdell (1960); Standart (1964); Delhay (1968a, b); Vernier & Delhay (1968); Slattery (1972). The mathematical roots can be found in the books by Weatherburn (1927); McConnell (1957) & Aris (1962).

2. TWO-PHASE SYSTEMS WITH SURFACE TENSION AND SURFACE PROPERTIES

2.1 The global balances

Consider a material volume $\mathcal{V}(t)$ cut by an interface $\mathcal{A}_i(t)$. This interface divides the material volume into two non-material volumes $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$, respectively limited by the surfaces $\mathcal{A}_1(t)$ and $\mathcal{A}_i(t)$, $\mathcal{A}_2(t)$ and $\mathcal{A}_i(t)$ (figure 1). The portion $\mathcal{A}_i(t)$ of the interface located inside the volume $\mathcal{V}(t)$ is limited by the curve $\mathcal{C}(t)$. As the volume $\mathcal{V}(t)$ is material, the surfaces $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$ are also material and we have at each point belonging to these surfaces:

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{v}_{\mathcal{A}} \cdot \mathbf{n} \tag{1}$$

where $\mathbf{v}_{\mathcal{A}} \cdot \mathbf{n}$ is the speed of displacement of the surface and \mathbf{v} the velocity of a particle.

The interface $\mathcal{A}_i(t)$ is not a material surface and a mass transfer may occur between the volumes $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ through $\mathcal{A}_i(t)$.

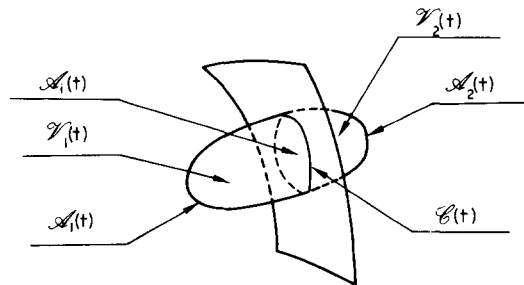


Figure 1. Diagram for global balances.

We assume that on the interface the tangential velocities are continuous, namely

$$\mathbf{v}'_1 = \mathbf{v}'_2, \quad [2]$$

but the temperatures are not necessarily continuous:

$$T_1 \neq T_2. \quad [3]$$

Balance of mass. The mass contained in the material volume is constant with time:

$$\frac{d}{dt} \int_{\mathcal{V}_1(t)} \rho_1 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{V}_2(t)} \rho_2 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{A}_i(t)} \rho_i d\mathcal{A} = 0 \quad [4]$$

where ρ_k is the density of phase k and ρ_i is the surface density (mass per unit of surface).

Balance of linear momentum. The external forces acting on the surface $\mathcal{A}_i(t)$ and its boundary $\mathcal{C}(t)$ are the external forces per unit of mass (gravity forces) and the line forces. In the three dimensional case, the elementary force acting on the surface element $d\mathcal{A}$ is:

$$\mathbf{n} \cdot \boldsymbol{\pi} d\mathcal{A}$$

where $\boldsymbol{\pi}$ is the stress tensor and \mathbf{n} the unit normal vector to the surface element. Similarly we could write that the force acting on the line element of the curve $\mathcal{C}(t)$ is:

$$\mathbf{N} \cdot \boldsymbol{\sigma} d\mathcal{C}$$

where $\boldsymbol{\sigma}$ is the surface stress tensor and \mathbf{N} the unit normal vector located in the tangent plane and directed outward the area $\mathcal{A}_i(t)$.

If we assume that the surface fluid is inviscid we can say that this force reduces to:

$$\mathbf{N} \sigma d\mathcal{C}$$

where σ is the surface tension (a scalar). Notice that this scalar has the same role as the pressure p (or more exactly $-p$) in the three dimensional case.

The balance of linear momentum can then be stated in the following way: The time-rate of change of the linear momentum of the moving material control volume $\mathcal{V}(t)$ is equal to the sum of the external forces acting on:

- the volumes $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$: external forces per unit of mass \mathbf{F}_1 and \mathbf{F}_2 .
- the interface $\mathcal{A}_i(t)$: external force \mathbf{F}_i per unit of mass.
- the surfaces $\mathcal{A}_1(t)$ and $\mathcal{A}_2(t)$: stresses $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$.
- the line $\mathcal{C}(t)$ surface tension σ .

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}_1(t)} \rho_1 \mathbf{v}_1 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{V}_2(t)} \rho_2 \mathbf{v}_2 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{A}_i(t)} \rho_i \mathbf{v}_p d\mathcal{A} &= \int_{\mathcal{V}_1(t)} \rho_1 \mathbf{F}_1 d\mathcal{V} \\ &+ \int_{\mathcal{V}_2(t)} \rho_2 \mathbf{F}_2 d\mathcal{V} + \int_{\mathcal{A}_i(t)} \rho_i \mathbf{F}_i d\mathcal{A} + \int_{\mathcal{A}_1(t)} \mathbf{n}_1 \cdot \boldsymbol{\pi}_1 d\mathcal{A} \\ &+ \int_{\mathcal{A}_2(t)} \mathbf{n}_2 \cdot \boldsymbol{\pi}_2 d\mathcal{A} + \oint_{\mathcal{C}(t)} \sigma \mathbf{N} d\mathcal{C}. \end{aligned} \quad [5]$$

Balance of angular momentum. The time rate of change of the angular momentum of the volume $\mathcal{V}(t)$ is equal to the resultant of the external torques acting on the volume $\mathcal{V}(t)$, the interface $\mathcal{A}_i(t)$ and their boundaries $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$ and $\mathcal{C}(t)$:

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}_1(t)} \mathbf{r} \times \rho_1 \mathbf{v}_1 \, d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{V}_2(t)} \mathbf{r} \times \rho_2 \mathbf{v}_2 \, d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{A}_i(t)} \mathbf{r} \times \rho_i \mathbf{v}_p \, da &= \int_{\mathcal{V}_1(t)} \mathbf{r} \times \rho_1 \mathbf{F}_1 \, d\mathcal{V} \\ &+ \int_{\mathcal{V}_2(t)} \mathbf{r} \times \rho_2 \mathbf{F}_2 \, d\mathcal{V} + \int_{\mathcal{A}_i(t)} \mathbf{r} \times \rho_i \mathbf{F}_i \, d\mathcal{A} \\ &+ \int_{\mathcal{A}_1(t)} \mathbf{r} \times (\mathbf{n} \cdot \boldsymbol{\pi}_1) \, d\mathcal{A} + \int_{\mathcal{A}_2(t)} \mathbf{r} \times (\mathbf{n}_2 \cdot \boldsymbol{\pi}_2) \, d\mathcal{A} \\ &+ \oint_{\mathcal{C}(t)} \mathbf{r} \times \sigma \mathbf{N} \, d\mathcal{C} \end{aligned} \tag{6}$$

where \mathbf{r} is the position vector.

Balance of total energy. The time rate of change of the total energy (kinetic energy and internal energy) of the volume $\mathcal{V}(t)$ is equal to the sum of:

- the power of the external forces acting on $\mathcal{V}(t)$, $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$, $\mathcal{A}_i(t)$, $\mathcal{C}(t)$,
- and the heat fluxes \mathbf{q}_1 , \mathbf{q}_2 , \mathbf{q}_i , entering the volume $\mathcal{V}(t)$ through $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$ and $\mathcal{C}(t)$.

$$\begin{aligned} \frac{d}{dt} \int_{\mathcal{V}_1(t)} \rho_1 \left(\frac{1}{2} v_1^2 + u_1 \right) \, d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{V}_2(t)} \rho_2 \left(\frac{1}{2} v_2^2 + u_2 \right) \, d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{A}_i(t)} \rho_i \left(u_i + \frac{1}{2} v_p^2 \right) \, d\mathcal{A} \\ = \int_{\mathcal{V}_1(t)} \rho_1 \mathbf{F}_1 \cdot \mathbf{v}_1 \, d\mathcal{V} + \int_{\mathcal{V}_2(t)} \rho_2 \mathbf{F}_2 \cdot \mathbf{v}_2 \, d\mathcal{V} \\ + \int_{\mathcal{A}_i(t)} \rho_i \mathbf{F}_i \cdot \mathbf{v}_p \, d\mathcal{A} + \int_{\mathcal{A}_i(t)} (\mathbf{n}_1 \cdot \boldsymbol{\pi}_1) \cdot \mathbf{v}_1 \, d\mathcal{A} \\ + \int_{\mathcal{A}_2(t)} (\mathbf{n}_2 \cdot \boldsymbol{\pi}_2) \cdot \mathbf{v}_2 \, d\mathcal{A} + \oint_{\mathcal{C}(t)} \sigma \mathbf{v}_p \cdot \mathbf{N} \, d\mathcal{C} \\ - \int_{\mathcal{A}_1(t)} \mathbf{q}_1 \cdot \mathbf{n}_1 \, d\mathcal{A} - \int_{\mathcal{A}_2(t)} \mathbf{q}_2 \cdot \mathbf{n}_2 \, d\mathcal{A} - \oint_{\mathcal{C}(t)} \mathbf{q}_i \cdot \mathbf{N} \, d\mathcal{C} \end{aligned} \tag{7}$$

where u_k and u_i are the internal energy per unit mass of phase k and interface, respectively. The normal component of \mathbf{v}_p is the speed of displacement of the interface and its tangential component is equal to the tangential components of the fluid velocities:

$$\mathbf{v}_p = (\mathbf{v}_i \cdot \mathbf{n}_k) \mathbf{n}_k + \mathbf{v}_t. \tag{8}$$

Entropy evolution. The first law of thermodynamics is expressed by the total energy balance [7]. The second law of thermodynamics is now formulated in terms of an inequality since the second law is an evolution law. Nevertheless, this inequality can be transformed into an equation by introducing an entropy generation term which must be positive for an irreversible evolution or equal to zero for a reversible one. According to the second law,

the time rate of change of the volume $\mathcal{V}(t)$ is equal to the sum of the influx of entropy due to conduction through $\mathcal{A}_1(t)$, $\mathcal{A}_2(t)$, $\mathcal{C}(t)$ and of the entropy sources in $\mathcal{V}_1(t)$, $\mathcal{V}_2(t)$ and on $\mathcal{A}_i(t)$:

$$\begin{aligned} & \frac{d}{dt} \int_{\mathcal{V}_1(t)} \rho_1 s_1 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{V}_2(t)} \rho_2 s_2 d\mathcal{V} + \frac{d}{dt} \int_{\mathcal{A}_i(t)} \rho_i s_i d\mathcal{A} + \int_{\mathcal{A}_1(t)} \frac{1}{T_1} \mathbf{q}_1 \cdot \mathbf{n}_1 d\mathcal{A} \\ & + \int_{\mathcal{A}_2(t)} \frac{1}{T_2} \mathbf{q}_2 \cdot \mathbf{n}_2 d\mathcal{A} + \oint_{\mathcal{C}(t)} \frac{1}{T_i} \mathbf{q}_i \cdot \mathbf{N} d\mathcal{C} \\ & = \int_{\mathcal{V}_1(t)} \Delta_1 d\mathcal{V} + \int_{\mathcal{V}_2(t)} \Delta_2 d\mathcal{V} + \int_{\mathcal{A}_i(t)} \Delta_i d\mathcal{A} \geq 0 \end{aligned} \quad [9]$$

where the equality occurs only for a reversible evolution. In this relation, s_k is the entropy per unit of mass of phase k , s_i the entropy per unit of mass of the interface, T_k the absolute temperature in phase k , T_i the absolute temperature on the interface, Δ_k the local entropy source per unit of volume and per unit of time in phase k and Δ_i the local entropy per unit area and per unit of time on the interface.

As we have, according to [9]:

$$\int_{\mathcal{V}_1(t)} \Delta_1 d\mathcal{V} + \int_{\mathcal{V}_2(t)} \Delta_2 d\mathcal{V} + \int_{\mathcal{A}_i(t)} \Delta_i d\mathcal{A} \geq 0$$

whatever $\mathcal{V}_1(t)$, $\mathcal{V}_2(t)$ and $\mathcal{A}_i(t)$ are, we get:

$$\Delta_k \geq 0 \quad [10]$$

$$\Delta_i \geq 0. \quad [11]$$

2.2 The primary jump conditions

Starting from the global balances of mass [4], linear momentum [5], angular momentum [6], total energy and from the entropy evolution [9], we derive the primary jump conditions by using the Leibniz rule (A1),* the surface Reynolds transport theorem (A2), the Gauss theorems, and the special forms of the latter (A3) and (A4). In that manner, we obtain a sum of two volume integrals and a surface integral. The volume integral furnishes the local partial differential equations valid in each phase, the surface integral furnishes the jump conditions valid on the interface only.

Mass jump condition.

$$\dot{m}_1 + \dot{m}_2 = \dot{m}_i \quad [12]$$

$$\text{with} \quad \dot{m}_k = \rho_k(\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}_k, \quad (k = 1, 2), \quad \dot{m}_i = d\rho_i/dt + \rho_i \nabla_s \cdot \mathbf{v}_p: \quad [13, 14]$$

Linear momentum jump condition.

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \dot{\mathbf{p}}_i \quad [15]$$

$$\text{with} \quad \dot{\mathbf{p}}_k = \dot{m}_k \mathbf{v}_k - \boldsymbol{\pi}_k \cdot \mathbf{n}_k, \quad (k = 1, 2) \quad [16]$$

$$\dot{\mathbf{p}}_i = \rho_i d\mathbf{v}_p/dt + \mathbf{v}_p \dot{m}_i - \rho_i \mathbf{F}_i - \nabla_s \sigma + (\nabla_s \cdot \mathbf{n}) \sigma \mathbf{n}. \quad [17]$$

* See Appendix.

Angular momentum jump condition. The derivation leads to the same condition as [15].

Total energy jump condition.

$$\dot{e}_1 + \dot{e}_2 = \dot{e}_i \quad [18]$$

with

$$\dot{e}_k = \dot{m}_k(u_k + \frac{1}{2}v_k^2) + \mathbf{q}_k \cdot \mathbf{n}_k - (\boldsymbol{\pi}_k \cdot \mathbf{n}_k) \cdot \mathbf{v}_k \quad [19]$$

$$\dot{e}_i = \rho_i \frac{d}{dt}(u_i + \frac{1}{2}v_p^2) + (u_i + \frac{1}{2}v_p^2)\dot{m}_i - \rho_i \mathbf{F}_i \cdot \mathbf{v}_p + \nabla_s \cdot \mathbf{q}_i - \nabla_s \cdot (\boldsymbol{\sigma} \mathbf{v}_i). \quad [20]$$

Entropy evolution.

$$\Delta_i = -\dot{m}_1 s_1 - \dot{m}_2 s_2 - \frac{1}{T_1} \mathbf{q}_1 \cdot \mathbf{n}_1 - \frac{1}{T_2} \mathbf{q}_2 \cdot \mathbf{n}_2 + \rho_i \frac{ds_i}{dt} + s_i \dot{m}_i + \nabla_s \cdot \frac{\mathbf{q}_i}{T_i} \geq 0. \quad [21]$$

2.3 The secondary local equations

Kinetic or mechanical energy equation. Multiplying the momentum equation [15] by the velocity vector \mathbf{v}_p , we get:

$$\begin{aligned} \dot{m}_1 \mathbf{v}_1 \cdot \mathbf{v}_p + \dot{m}_2 \mathbf{v}_2 \cdot \mathbf{v}_p - (\boldsymbol{\pi}_1 \cdot \mathbf{n}_1) \cdot \mathbf{v}_p - (\boldsymbol{\pi}_2 \cdot \mathbf{n}_2) \cdot \mathbf{v}_p - \rho_i \frac{d}{dt}(\frac{1}{2}v_p^2) + \rho_i \mathbf{F}_i \cdot \mathbf{v}_p + \mathbf{v}_p \cdot \nabla_s \boldsymbol{\sigma} \\ - (\nabla_s \cdot \mathbf{n}) \boldsymbol{\sigma} \mathbf{v}_p \cdot \mathbf{n} - v_p^2 \dot{m}_i = 0. \end{aligned} \quad [22]$$

Internal energy jump condition. Subtracting the mechanical energy jump condition [22] from the total energy jump condition [18] we obtain:

$$\begin{aligned} \dot{m}_1(u_1 + \frac{1}{2}v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_p) + \dot{m}_2(u_2 + \frac{1}{2}v_2^2 - \mathbf{v}_2 \cdot \mathbf{v}_p) + \mathbf{q}_1 \cdot \mathbf{n}_1 + \mathbf{q}_2 \cdot \mathbf{n}_2 - (\boldsymbol{\pi}_1 \cdot \mathbf{n}_1) \cdot (\mathbf{v}_1 - \mathbf{v}_p) \\ - (\boldsymbol{\pi}_2 \cdot \mathbf{n}_2) \cdot (\mathbf{v}_2 - \mathbf{v}_p) - \rho_i \frac{du_i}{dt} - (u_i - \frac{1}{2}v_p^2)\dot{m}_i \\ - \nabla_s \cdot \mathbf{q}_i + \boldsymbol{\sigma} \nabla_s \cdot \mathbf{v}_p = 0. \end{aligned} \quad [23]$$

Enthalpy jump condition. Introduce the following quantities:

—the enthalpies i_k defined by:

$$i_k = u_k + \frac{P_k}{\rho_k} \quad (k = 1, 2) \quad [24]$$

—the deviatoric stress tensors $\boldsymbol{\tau}_k$ defined by:

$$\boldsymbol{\pi}_k = -p_k \mathcal{U} + \boldsymbol{\tau}_k \quad (k = 1, 2) \quad [25]$$

—the surface enthalpy defined by:

$$i_i = u_i - \frac{\boldsymbol{\sigma}}{\rho_i}. \quad [26]$$

Taking into account equations [24]–[26] and [14], the internal energy jump condition can be changed into the following form:

$$\begin{aligned} \dot{m}_1(i_1 + \frac{1}{2}v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_p) + \dot{m}_2(i_2 + \frac{1}{2}v_2^2 - \mathbf{v}_2 \cdot \mathbf{v}_p) + \mathbf{q}_1 \cdot \mathbf{n}_1 + \mathbf{q}_2 \cdot \mathbf{n}_2 - (\boldsymbol{\tau}_1 \cdot \mathbf{n}_1) \cdot (\mathbf{v}_1 - \mathbf{v}_p) \\ - (\boldsymbol{\tau}_2 \cdot \mathbf{n}_2) \cdot (\mathbf{v}_2 - \mathbf{v}_p) - \rho_i \frac{di_i}{dt} - \frac{d\sigma}{dt} - (i_i - \frac{1}{2}v_p^2)\dot{m}_i - \nabla_s \cdot \mathbf{q}_i = 0. \end{aligned} \quad [27]$$

Entropy jump condition. In order to transform the enthalpy jump condition [27] into an entropy jump condition, we have to introduce the free enthalpies defined by:

$$g_k = i_k - T_k s_k \quad (k = 1, 2) \quad [28]$$

$$g_i = i_i - T_i s_i. \quad [29]$$

The Gibbs equation, written for the interface, has the following form:

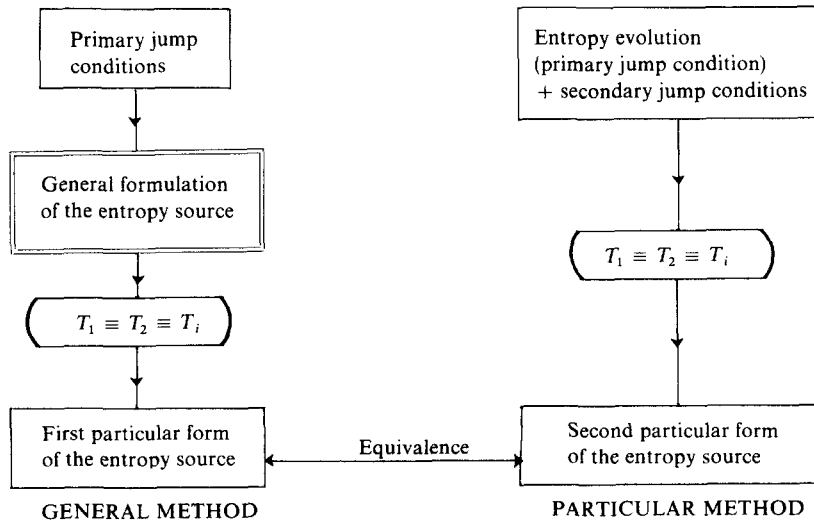
$$\frac{du_i}{dt} = T_i \frac{ds_i}{dt} - \frac{\sigma}{\rho_i^2} \frac{d\rho_i}{dt}. \quad [30]$$

Taking into account equations [28]–[30], the enthalpy jump condition [27] becomes:

$$\begin{aligned} \dot{m}_1(T_1 s_1 + g_1 + \frac{1}{2}v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_p) + \dot{m}_2(T_2 s_2 + g_2 + \frac{1}{2}v_2^2 - \mathbf{v}_2 \cdot \mathbf{v}_p) + \mathbf{q}_1 \cdot \mathbf{n}_1 + \mathbf{q}_2 \cdot \mathbf{n}_2 \\ - (\boldsymbol{\tau}_1 \cdot \mathbf{n}_1) \cdot (\mathbf{v}_1 - \mathbf{v}_p) - (\boldsymbol{\tau}_2 \cdot \mathbf{n}_2) \cdot (\mathbf{v}_2 - \mathbf{v}_p) \\ - \rho_i T_i \frac{ds_i}{dt} - (T_i s_i + g_i - \frac{1}{2}v_p^2)\dot{m}_i - \nabla_s \cdot \mathbf{q}_i = 0. \end{aligned} \quad [31]$$

2.4. The interfacial entropy source

In the case of single-phase flow, the local entropy source is obtained from the secondary local equations. More precisely (Slattery 1972), the entropy evolution equation (primary



equation) is combined with the entropy equation (secondary equation deduced from the internal energy equation through the Gibbs relation). Now, as far as the jump conditions are concerned, we follow a different method which is represented in the diagram.

In the *general method*, we start from the following primary jump conditions:

- mass [12],
- momentum [15],
- total energy [18],
- entropy evolution [21].

By combining these jump conditions we get the *general* form of the local interface entropy source:

$$\begin{aligned} \Delta_i = & \dot{m}_1 \left[\frac{1}{T_2} \left(\frac{p_2}{\rho_2} - g_2 + \frac{1}{2} v_2^2 \right) - \frac{1}{T_1} \left(\frac{p_1}{\rho_1} - g_1 + \frac{1}{2} v_1^2 \right) \right] + \dot{\mathbf{p}}_1 \cdot \left(\frac{1}{T_1} \mathbf{v}_1 - \frac{1}{T_2} \mathbf{v}_2 \right) \\ & + \dot{e}_1 \left(\frac{1}{T_2} - \frac{1}{T_1} \right) + \dot{m}_i \left[\frac{1}{T_i} \left(\frac{1}{2} v_p^2 - g_i \right) - \frac{1}{T_2} \left(\frac{p_2}{\rho_2} - g_2 + \frac{1}{2} v_2^2 \right) \right] \\ & + \dot{\mathbf{p}}_i \cdot \left(\frac{1}{T_2} \mathbf{v}_2 - \frac{1}{T_i} \mathbf{v}_p \right) + \dot{e}_i \left(\frac{1}{T_i} - \frac{1}{T_2} \right) + \mathbf{q}_i \cdot \nabla_s \left(\frac{1}{T_i} \right) \end{aligned} \quad [32]$$

or, in a more symmetrical form:

$$\begin{aligned} \Delta_i = & \dot{m}_1 \left[\frac{1}{T_i} \left(\frac{1}{2} v_p^2 - g_i \right) - \frac{1}{T_1} \left(\frac{1}{2} v_1^2 - g_1 + \frac{p_1}{\rho_1} \right) \right] \\ & + \dot{m}_2 \left[\frac{1}{T_i} \left(\frac{1}{2} v_p^2 - g_i \right) - \frac{1}{T_2} \left(\frac{1}{2} v_2^2 - g_2 + \frac{p_2}{\rho_2} \right) \right] + \dot{\mathbf{p}}_1 \cdot \left(\frac{1}{T_1} \mathbf{v}_1 - \frac{1}{T_i} \mathbf{v}_p \right) \\ & + \dot{\mathbf{p}}_2 \cdot \left(\frac{1}{T_2} \mathbf{v}_2 - \frac{1}{T_i} \mathbf{v}_p \right) + \dot{e}_1 \left(\frac{1}{T_i} - \frac{1}{T_1} \right) + \dot{e}_2 \left(\frac{1}{T_i} - \frac{1}{T_2} \right) \\ & + \mathbf{q}_i \cdot \nabla_s \left(\frac{1}{T} \right). \end{aligned} \quad [33]$$

The first particular form is obtained by putting: $T_1 \equiv T_2 \equiv T_i$ in the general expression [32]. We get:

$$\begin{aligned} T_i \Delta_i = & \dot{m}_1 \left[\left(\frac{1}{2} v_2^2 + \frac{p_2}{\rho_2} - g_2 \right) - \left(\frac{1}{2} v_1^2 + \frac{p_1}{\rho_1} - g_1 \right) \right] + \dot{\mathbf{p}}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_2) + T_i \mathbf{q}_i \cdot \nabla_s \left(\frac{1}{T_i} \right) \\ & + \dot{m}_i \left[\left(\frac{1}{2} v_p^2 - g_i \right) - \left(\frac{p_2}{\rho_2} - g_2 + \frac{1}{2} v_2^2 \right) \right] + \dot{\mathbf{p}}_i \cdot (\mathbf{v}_2 - \mathbf{v}_p) \end{aligned} \quad [34]$$

or:

$$T_i \Delta_i = \dot{m}_1 \left[\left(\frac{1}{2} v_p^2 - g_i \right) - \left(\frac{1}{2} v_1^2 + \frac{p_1}{\rho_1} - g_1 \right) \right] + \dot{m}_2 \left[\left(\frac{1}{2} v_p^2 - g_i \right) - \left(\frac{1}{2} v_2^2 + \frac{p_2}{\rho_2} - g_2 \right) \right]$$

$$+ \dot{\mathbf{p}}_1 \cdot (\mathbf{v}_1 - \mathbf{v}_p) + \dot{\mathbf{p}}_2 \cdot (\mathbf{v}_2 - \mathbf{v}_p) + T_i \mathbf{q}_i \cdot \nabla_s \left(\frac{1}{T_i} \right). \quad [35]$$

In the *particular method*, we start from the primary jump condition on the entropy [21] and from the secondary jump condition on the entropy [31], after assuming that, on the interface $T_1 \equiv T_2 \equiv T_i$, we obtain:

$$\begin{aligned} T_i \Delta_i = & \dot{m}_1 \left[g_1 - g_i + \frac{1}{2} (\mathbf{v}_1 - \mathbf{v}_p)^2 - \frac{1}{\rho_1} (\boldsymbol{\tau}_1 \cdot \mathbf{n}_1) \cdot \mathbf{n}_1 \right] \\ & + \dot{m}_2 \left[g_2 - g_i + \frac{1}{2} (\mathbf{v}_2 - \mathbf{v}_p)^2 - \frac{1}{\rho_2} (\boldsymbol{\tau}_2 \cdot \mathbf{n}_2) \cdot \mathbf{n}_2 \right] \\ & + T_i \mathbf{q}_i \cdot \nabla_s \left(\frac{1}{T_i} \right) \end{aligned} \quad [36]$$

which can be shown to be identical to [35].

We could have derived an entropy source without taking into account the surface properties, keeping only the surface tension effect. In this case, the analogous expression of [36] can be shown to be (Delhaye 1974):

$$\begin{aligned} T_i \Delta_i = & \dot{m}_1 \left[g_1 + \frac{1}{2} v_1^2 - \mathbf{v}_1 \cdot \mathbf{v}_p - \frac{1}{\rho_1} (\boldsymbol{\tau}_1 \cdot \mathbf{n}_1) \cdot \mathbf{n}_1 \right] \\ & + \dot{m}_2 \left[g_2 + \frac{1}{2} v_2^2 - \mathbf{v}_2 \cdot \mathbf{v}_p - \frac{1}{\rho_2} (\boldsymbol{\tau}_2 \cdot \mathbf{n}_2) \cdot \mathbf{n}_2 \right] \\ & + \sigma \nabla_s \cdot \mathbf{v}_p \geq 0. \end{aligned} \quad [37]$$

Equation [36] cannot be reduced to [37] because of the surface divergence term in the latter. Let us see the kind of inconsistency we can get if we do not take into consideration the material properties of the interface to derive the entropy source [37].

Consider a gas sphere in a liquid. The radius is a function of time. If there is no mass transfer through the interface and if there is no temperature jump across the interface, the entropy source is equal to zero [36]. But following [37] we have:

$$\Delta_i = \frac{1}{T_i} \sigma \nabla_s \cdot \mathbf{v}_p. \quad [38]$$

If the gas sphere has a purely radial motion, we obtain:

$$\Delta_i = \frac{2\sigma \dot{R}}{T_i R} \quad [39]$$

where \dot{R} is the time derivative of R . Then, the entropy source can be negative if the radius decreases, which violates the second law of thermodynamics.

The important conclusion is that it is necessary either to account simultaneously for the surface tension *and* the surface material properties or neither one. We have to point out that it is normal to get some inconsistency, if the surface tension is introduced without

the surface properties for the surface tension, and the surface properties are connected through [26] and Gibbs relation [30]. Consequently, it seems logical that the inconsistency appears with the enthalpy jump condition.

In conclusion, if we write the jump condition and the entropy source with the surface tension only, the surface density ρ_i and the surface heat flux \mathbf{q}_i must vanish.

3. CONCLUSIONS

In this paper, we systematically formulated the jump conditions and the interface entropy source for a two-phase system in the absence of electromagnetic phenomena. Surface tension and surface properties were taken into account.

The results can be reduced to some special cases previously obtained (see for example Bornhorst & Hatsopoulos 1967).

A complete set of jump conditions is provided, which has to be used in the time- or space-average formulations of two-phase flow models (Ishii 1974).

It has been shown that it was necessary to take into account simultaneously the surface tension and the surface properties to derive consistent jump conditions and entropy sources.

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APPENDIX

The Leibniz rule

Let us consider a geometric volume $\mathcal{V}(t)$ which is moving in space (figure A1). This volume $\mathcal{V}(t)$ is bounded by a closed surface $\mathcal{A}(t)$. At a given point belonging to this surface $\mathcal{A}(t)$ let \mathbf{n} be the unit normal vector, outwardly directed. The speed of displacement of the surface at this point is $\mathbf{v}_{\mathcal{A}} \cdot \mathbf{n}$. Let $f(x, y, z, t)$ be a function of point $M(x, y, z)$ and of time t .

The Leibniz rule says that:

$$\frac{d}{dt} \int_{\mathcal{V}(t)} f(x, y, z, t) d\mathcal{V} = \int_{\mathcal{V}(t)} \frac{\partial f}{\partial t} d\mathcal{V} + \oint_{\mathcal{A}(t)} f \mathbf{v}_{\mathcal{A}} \cdot \mathbf{n} d\mathcal{A}. \quad [\text{A1}]$$

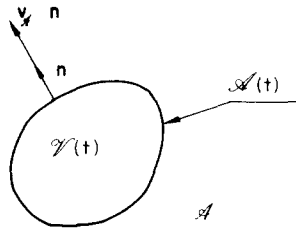


Figure A1. Diagram for Leibniz rule.

It has to be noticed that the volume $\mathcal{V}(t)$ is a geometric volume and not necessarily a material volume. The Leibniz rule is a purely kinematic theorem.

The surface Reynolds transport theorem (Aris 1962)

The material area $\mathcal{A}_i(t)$ (figure A2) occupies a new position $\mathcal{A}_i(t + \Delta t)$ at time $t + \Delta t$ and the point M shifts to M' . Let f_i be a function defined on the surface and only on the surface. The Reynolds transport theorem for the material area $\mathcal{A}_i(t)$ says that:

$$\frac{d}{dt} \int_{\mathcal{A}_i(t)} f_i d\mathcal{A} = \int_{\mathcal{A}_i(t)} \left(\frac{df_i}{dt} + f_i \nabla_s \cdot \mathbf{v}_p \right) d\mathcal{A} \quad [\text{A2}]$$

where we have:

$$\frac{df_i}{dt} = \lim_{\Delta t \rightarrow 0} \frac{f_i(M') - f_i(M)}{\Delta t}$$

$$\mathbf{v}_p = \lim_{\Delta t \rightarrow 0} \frac{\overrightarrow{MM'}}{\Delta t}.$$

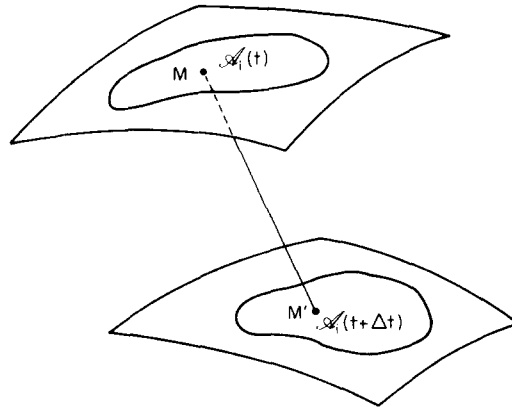


Figure A2. Diagram for Reynolds transport theorem.

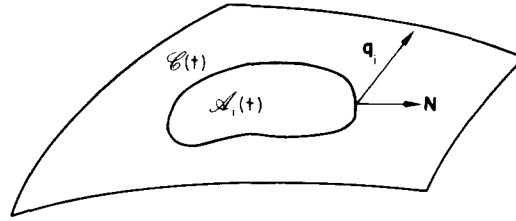


Figure A3. Diagram for Gauss theorems.

The Gauss theorems for a surface (Aris 1962)

In a three-dimensional space, the Gauss theorems enable the transformation of a surface integral into a volume integral. Similarly, in a two-dimensional space, the Gauss theorems enable the transformation of a line integral into a surface integral:

$$\oint_{C(t)} f_i \mathbf{N} \, d\mathcal{C} = \int_{A_i(t)} [\nabla_s f_i - (\nabla_s \cdot \mathbf{n}) f_i \mathbf{n}] \, d\mathcal{A} \tag{A3}$$

$$\oint_{C(t)} \mathbf{q}_i \cdot \mathbf{N} \, d\mathcal{C} = \int_{A_i(t)} \nabla_s \cdot \mathbf{q}_i \, d\mathcal{A}. \tag{A4}$$

\mathbf{N} is the unit normal vector at a given point belonging to the curve $\mathcal{C}(t)$, boundary of $A_i(t)$ (figure A3). The vector \mathbf{N} is directed outward the area $A_i(t)$ and located in the tangent plane. ∇_s is the surface del operator (McConnell 1957; Weatherburn 1927).

\mathbf{q}_i is a vector tangent to the surface $A_i(t)$. The surface divergence $\nabla_s \cdot \mathbf{n}$ is equal to twice the mean curvature.

Sommaire—Cet article a pour objet la mécanique des systèmes diphasiques lorsque l'on prend en compte la tension interfaciale et les propriétés thermodynamiques de l'interface.

Les formes intégrales des bilans de masse, de quantités de mouvement linéaire et angulaire, d'énergie totale et d'entropie sont posées *a priori*.

Partant de ces lois globales, on calcule les conditions d'interface et la source d'entropie à l'interface.

Auszug—Die Kontinuumsmechanik von Zweiphasensystemen, unter Berücksichtigung von Oberflächenspannung und Oberflächeneigenschaften, wird behandelt. Bilanzgleichungen in integraler Form werden fuer die folgenden Groessen angegeben: Masse, Bewegungsgroesse, Drall, Gesamtenergie und Entropie. Ausgehend von diesen integralen Bilanzgleichungen werden die Sprungbedingungen und die Entropiequelle in der Zwischenschicht abgeleitet.

Резюме—В статье рассматривается механика континуума двухфазных систем, включающих поверхностное натяжение и поверхностные характеристики. Интегральные формы соотношений баланса даны для следующих величин: массы, линейного момента количества движения, углового момента количества движения, общей энергии и энтропии. Исходя из этих интегральных соотношений баланса, условия скачка и источник энтропии на поверхности раздела получаются.